

# MAXENT and the Tsallis Parameter

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## Abstract

The nonextensive entropic measure proposed by Tsallis introduces a parameter,  $q$ , which is not defined but rather must be determined. The value of  $q$  is typically determined from a piece of data and then fixed over the range of interest. On the other hand, from a phenomenological viewpoint, there are instances in which  $q$  cannot be treated as a constant. We present two distinct approaches for determining  $q$  depending on the form of the equations of constraint for the particular system. In the first case the equations of constraint for an operator  $O$  can be written as  $Tr[F^q O] = C$ , where  $C$  may be an explicit function of the distribution function,  $F$ . In this case one can solve an equivalent MAXENT problem which yields  $q$  as a function of the corresponding Lagrange Multiplier. As an illustration the exact solutions to the static Generalized Fokker-Planck Equation (GFP) are obtained from MAXENT. As in the case where  $C$  is a constant if  $q$  is treated as a variable within the MAXENT framework, the entropic measure is maximized for all values of  $q$  trivially. Therefore  $q$  must be determined from existing data. In the second case an additional equation of constraint exists which cannot be brought into the above form. In this case the additional equation of constraint may be used to determine the fixed value of  $q$ .

Since its introduction the Tsallis entropy[1–5] has increasingly been utilized as the entropic measure in the Maximum Entropy (MAXENT) calculations[6]. The choice of the Tsallis parameter,  $q$ , which is not defined *a priori*[7, 8] can yield in the limit  $q \rightarrow 1$  the Boltzmann-Gibbs (BG) entropy as well as a family of fractal entropies ( $q \neq 1$ ). It has been argued that perhaps this parameter should be constant, if not universally, at least for classes of dynamical systems[9–11] and attempts have been made to set limits on the value of this parameter [12]. Within the framework of the MAXENT approach the question arises as to how one should determine  $q$ .

For a classical system the stationary state distribution function,  $F$ , can be obtained from the classical Tsallis entropy

$$S_q = \frac{1}{q-1}(F - F^q) \quad (1)$$

via the MAXENT equation[4]

$$\delta_F S_q = 0 \quad (2)$$

along with the equation of constraint

$$Tr[F^q O] = C \quad (3)$$

The solution of these equations yields the classical Tsallis distribution given by

$$F^q(x) = D[1 - \beta(1 - q)O(x)]^{\frac{1}{1-q}}, \quad (4)$$

where  $D$  is a constant. We consider two cases.

In the first case all of the equations of constraint are of the form of equation(3). We consider the following non-linear one dimensional Generalized Fokker-Planck (GFP) equation[13]

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial x}\{K(x)F\} + \frac{1}{2}Q\frac{\partial^2[F^{2-q}]}{\partial x^2}, \quad (5)$$

where  $F$  is the distribution function,  $Q$  is the diffusion coefficient, and  $K(x)$  is the drift coefficient which determines the potential:

$$V(x) = -\int_{x_0}^x K(x)dx. \quad (6)$$

The particular power  $q = 2$  is chosen in accordance with the quite general discussion of the generalized Bogulubov inequalities[14] which points out that systems which obey Tsallis

statistics exhibit abrupt changes at  $q = 2$ . An exact solution (both static and time dependent) of the GFP equations has been found and is shown under certain circumstances to be equivalent to the Tsallis classical distribution functions[4].

In the static case we formulate and solve the equivalent MAXENT problem where the  $C$  is an explicit function of the distribution functions. In the GFP case one has

$$K(x)F = 1/2Q \frac{\partial F^{2-q}}{\partial x} \quad (7)$$

$$\int^x K F dx = Q/2(F^{2-q}(x) - F^{2-q}(x_0)) \quad (8)$$

Integrating by parts ( $K = -\frac{\partial V}{\partial x}$  and  $V(x_0) = 0$ )

$$V(x)F(x) = -Q/2(F^{2-q}(x) - F^{2-q}(x_0)) + \int V \frac{\partial F}{\partial x} \quad (9)$$

or

$$tr[VF^q] = tr[F^{q-1}(\int V \frac{\partial F}{\partial x} dx - Q/2(F^{2-q}(x) - F^{2-q}(x_0)))] \quad (10)$$

Now eq(7) is solved by the solution given in the Plastinos paper

$$F(x) = D[1 - \beta(1 - q)V(x)]^{\frac{1}{1-q}}. \quad (11)$$

One can verify by substitution that Eq.(4) is a solution to Eq. (7) provided

$$\beta = \frac{2}{Q} [\frac{D^{q-1}}{2-q}] \quad (12)$$

With  $D=1$  the solution is the same as that obtained from the above MAXENT equations (with eq(10) as the equation of constraint). Again the equation of constraint is only satisfied if  $\beta$  is given by equation(12) (with  $D=1$ ). Note in this case there is no solution if  $V(x)$  is a constant.

Generally the  $C$  in the equation of constraint is a constant rather than an explicit function of the distribution functions. In order to simultaneously determine  $q$  it has been suggested [5] that an additional equation

$$\frac{\partial S}{\partial q}|_{\beta} = 0 \quad (13)$$

must be solved. However it should be noted that this equation can be re-written as

$$\begin{aligned} \frac{\partial S}{\partial q}|_{\beta} &= \frac{\partial S}{\partial F}|_{\beta} \frac{\partial F}{\partial q}|_{\beta} \\ &= 0. \end{aligned}$$

Since Tsallis distribution functions (equation(4)) are the solution of equations(2) and (3) the above equation is trivially satisfied for any value of  $q$ . Hence  $q$  cannot be determined in this manner and one has no choice but to determine  $q$  from the existing data. For the practitioners this has been accepted de facto and calculations involving the Tsallis entropy have generally used one piece of data to determine the value of  $q$  which is then fixed over the range of interest[12].

In the second case an additional equation of constraint exists which cannot be brought into the form of equation (3). Such is the case in obtaining the distribution functions of the finite temperature BCS equations with the Tsallis single particle entropic measure. Aside from the self consistency requirements of the single quasi-particle energies, which are needed, the determination of the distribution functions follows from equation(2) along with the appropriate equations of constraint. The existence of a critical point where the gap vanishes yields an additional equation of constraint not of the form given by equation (3). As has been shown recently this, can be used to determine  $q$ [15, 16].

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